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# Iterative Approximation of Fixed Points for a New Class of Generalized Suzuki-Type Nonexpansive Mappings

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**Abstract**
We introduce a new class of generalized Suzuki-type nonexpansive mappings and prove strong convergence results to fixed points of the mappings. Our results generalize many important results in the literature.



## Introduction

Let $E$ be a Banach space and let $K$ be a nonempty subset of $E$. A mapping $T : K \to K$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. The set of fixed points of $T$ is denoted by $F(T)$,i.e., $F(T) = x \in K : Tx = x$. It is well known that if $E$ is uniformly convex and $K$ is bounded, closed and convex subset of $E$, then $F(T)$ is nonempty. Many authors (see e.g [1] - [25] and their references)have studied fixed point results for nonexpansive mappings and their generalizations. The map $T$ is called quasi-nonexpansive if $\|Tx - p\| \leq \|x - p\|$ for all $x \in K,\ p \in F(T)$.
Recently, Suzuki [21] introduced a weaker class of contractions and proved the following result:

**Theorem 1 (see [21]):** Let $\theta : [0, 1) \to (\frac{1}{2}, 1]$ be defined by

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq (\frac{\sqrt{5}-1}{2}), \\ (1-r)r^{-2} & \text{if } (\frac{\sqrt{5}-1}{2}) \leq r \leq 2^{\frac{-1}{2}}, \\ (1+r)^{-1} & \text{if } 2^{\frac{-1}{2}} \leq r < 1 \end{cases}$$

Then for a metric space $(X, d)$, the following are equivalent:
(i) $X$ is complete
(ii) There exists $r \in (0, 1)$ such that every mapping $T$ on $X$ satisfying
$\theta(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq rd(x, y),\ \forall x, y \in X$, has a fixed point.

More recently, using the idea of theorem 1, Suzuki (see [22]), defined a very natural condition satisfied by a class of mappings, known as condition $(C)$, as follows:

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**Definition 1 (see [22]):** Let $T$ be a mapping on a subset $C$ of a Banach space $X$. Then $T$ is said to satisfy *condition*$(C)$ if
$(C)$ $\frac{1}{2}||x - Tx|| \leq ||x - y||$ implies $||Tx - Ty|| \leq ||x - y||, \ \forall x, y \in C$.
The author showed that this condition is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness (see [22]). Furthermore, the author proved weak and strong convergence results for mappings satisfying *condition*$(C)$. More precisely, the author stated and proved the following theorems:

**Theorem 2 (see [22]):** Let $T$ be a mapping on a compact convex subset $C$ of a Banach space $X$. Assume that $T$ satisfies *condition (C)*. Define the sequence $\{x_n\}$ in $C$ by $x_1 \in C$ and

$$x_{n+1} = \lambda T x_n + (1 - \lambda) x_n$$

for $n \in N$, where $\lambda$ is a real number belonging to $[\frac{1}{2}, 1)$. Then $\{x_n\}$ converges strongly to a fixed point of $T$.

**Theorem 3(see [22]):** Let $T$ be a mapping on a weakly compact convex subset $C$ of a Banach space $X$ with the Opial property.Assume that $T$ satisfies *condition (C)*. Define the sequence $\{x_n\}$ in $C$ by $x_1 \in C$ and

$$x_{n+1} = \lambda T x_n + (1 - \lambda) x_n$$

for $n \in N$, where $\lambda$ is a real number belonging to $[\frac{1}{2}, 1)$. Then $\{x_n\}$ converges weakly to a fixed point of $T$.

**Theorem 4(see [22]):** Let $C$ be a weakly convex subset of a $UCED$ Banach space. Let $S$ be a family of commuting mappings on $C$ satisfying *condition (C)*. Then $S$ has a common fixed point.
The purpose of this paper is to introduce a class of mappings that satisfy a condition weaker than *condition (C)*. Furthermore, we prove strong convergence results to fixed points of our class of mappings, in certain Banach spaces. We give the following definition:

**Defintion 2:** Let $T$ be a mapping on a subset $C$ of a Banach space $X$, with $F(T) \neq \emptyset$. Then $T$ is said to satisfy *condition*$(C^*)$ if

$(C^*)$ $\frac{1}{2}||x - Tx|| \leq ||x - p||$ implies $||Tx - p \leq ||x - p||, \forall x \in C, \ p \in F(T)$

All quasi-nonexpansive mappings satisfy *condition*$(C^*)$. It is also obvious that any mapping $T$ that satisfies *condition*$(C)$ and possesses a non-empty fixed point set, satisfies *condition*$(C^*)$. We now present an example to show that there exist mappings which satisfy condition *condition*$(C^*)$ but do not satisfy *condition*$(C)$. Hence *condition*$(C^*)$ is weaker than *condition*$(C)$.

**Example:** Define $T : [-\pi,\ \pi] \to [-\pi,\ \pi]$ by $Tx = xcosx$. Clearly, $F(T) = \{0\}$. We have

$$\begin{aligned} \frac{1}{2}||x - Tx|| &= \frac{1}{2}|x - xcosx| \\ &= \frac{1}{2}|x||1 - cosx| \\ &\leq |x| \\ &= ||x|| \end{aligned}$$

Consider $x = \frac{\pi}{2}$ and $y = \pi$. We have
$||x - y|| = \frac{\pi}{2}$. Hence $\frac{1}{2}||x - Tx|| \leq ||x - y||$. However,
$||Tx - Ty|| = |\frac{\pi}{2}cos\frac{\pi}{2} - \pi cos\pi| = |\pi| > \frac{\pi}{2} = ||x - y||$. Hence, $T$ does not satisfy $condition(C)$.
On the other hand, we have

$$\begin{aligned} \frac{1}{2}||x - Tx|| &= \frac{1}{2}|x - xcosx| \\ &= \frac{1}{2}|x||1 - cosx| \\ &\leq |x| \\ &= ||x - p|| \end{aligned}$$

Furthermore,
$||Tx - p|| = |xcosx - 0| \leq |x| = ||x - p||$. Hence, $T$ satisfies $condition(C^*)$.
In [22], Suzuki noted that $condition(C)$ is weaker than nonexpansiveness. It is now obvious from the foregoing that $condition(C^*)$ is weaker that $condition(C)$ which is weaker than nonexpansiveness.

**Preliminaries**

Throughout this work we shall denote the set of natural numbers by $N$.

**Definition 3 (see [22]):** Banach space $X$ is said to be strictly convex if $\|x + y\| < 2$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$.

**Definition 4 (see [25]):** Let $X$ be a Banach space and $C$ be a subset of $X$. A mapping $T : C \to C$ is called demi-compact if whenever $\{x_n\}$ is a bounded sequence in the domain of $T$ such that $\{x_n - Tx_n\}$ converges strongly, then $\{x_n\}$ has a subsequence which converges strongly.

**Lemma 1(see e.g [23] ):** A Banach space $X$ is said to be uniformly convex if and only if there exist a continuous strictly increasing and convex function $g : [0,\ \infty) \to [0,\ \infty)$ satisfying $g(0) = 0$, such that

$$||\lambda x - (1-\lambda)y||^p \leq \lambda||x||^p + (1-\lambda)||y|^p - w_p(\lambda)g(||x - y||)$$

for all $x, y \in B_r := \{x \in X : ||x|| \leq r\},\ \lambda \in [0, 1]$ with $w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$, for $p>1,\ r>0$.

We now present some important lemmas, which will be useful in the sequel.

**Lemma 2 (see e.g [24]):** Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1+\delta_n)a_n + b_n,$$

for all $n \geq 1$. If $\sum \delta_n < \infty$ and $\sum b_n < \infty$, then $\lim a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim a_n = 0$.

**Lemma 3:** Let $T$ be a mapping on a closed subset $C$ of a Banach space $E$. Assume that $T$ satisfies condition $(C^*)$. Then $F(T)$ is closed. Moreover, if E is strictly convex and $C$ is convex, then $F(T)$ is also convex.

**Proof:** We want to prove the following:

1 . F(T) is closed; i.e. for $x_n \in F$,$x_n \to x$ then $x \in F(T)$

2 . F(T) is convex when $C$ is convex and $X$ is strictly convex.

Let $\{x_n\}$ be a sequence in $F(T)$ converging to some point $x \in C$. By condition $(C^*)$ we have
$\left(\frac{1}{2}\right) \|x_n - Tx_n\| = 0 \leq \|x_n - x\|$ for $n \in N$,
implies $\|Tx - x_n\| \leq \|x_n - x\|$ for $x \in C$ and $x_n \in F(T)$
It follows that $\limsup_{n\to\infty} \|Tx - x_n\| \leq \limsup_{n\to\infty} \|x_n - x\| = 0$

This implies $\limsup_{n\to\infty} \|Tx - x_n\| = 0$. Hence, $\limsup_{n\to\infty}\{Tx - x_n\} = 0$,
which shows that $\{x_n\}$ also converges to $Tx$. By the uniqueness of limits we have that $Tx = x$. Therefore $F(T$ is closed.
Next, by assuming that X is strictly convex and $C$ is convex, we show that $F(T)$ is convex. Given $\lambda \in (0,1)$ and $a, b \in F(T)$ with $a \neq b$, let $z = \lambda a + (1-\lambda)\, b \in C$. Then $F(T)$ is convex if $Tz = z$. Let
$a = \lambda a + (1-\lambda)\, a$
$b = \lambda b + (1-\lambda)\, b$
$z = \lambda a + (1-\lambda)\, b$
Since $X$ is strictly convex, there exists $s \in [0,1]$ such that $Tz = sa + (1-s)b$. Therefore, to show that $z \in F(T)$ ,i.e $Tz = z$, it suffices to show that $\lambda = s$. Since
Let $a = sa + (1-s)a$, then
$Tz - a = (1-s)(b-a)$. Therefore

$$\|a - Tz\| = (1-s)\|a-b\| \tag{1}$$

Since $T$ satisfies condition $C^*$, we have $\|a - Tz\| \leq \|a - z\|$ for $z \in C$ and $a \in F(T)$.
This implies

$$(1-s)\|a-b\| \leq \|a-z\| \tag{2}$$

Now, let $a = \lambda a + (1-\lambda)\, a$. Then
$a - z = (1-\lambda)\,(a-b)$. This yields

$$\|a-z\| = (1-\lambda)\,\|a-b\| \tag{3}$$

Substituting for $\|a-z\|$ in (2), we have

$$(1-s)\|a-b\| \leq (1-\lambda)\,\|a-b\| \tag{4}$$

Next, Observe that
$b = bs + (1-s)b$, so that
$Tz - b = s(a-b)$. This implies

$$\|Tz-b\| = s\|a-b\| \tag{5}$$

Also,
$z - b = \lambda(b-a)$, so that
$||z-b|| = \lambda||b-a||$

Furthermore,
$||Tz-b|| \leq ||z-b||$, by condition $C^*$. This implies

$$s\|b-a\| \leq \lambda\|b-a\| \tag{6}$$

From (4) and (6), we have $s = \lambda$. This completes the proof.

**Lemma 4:** Let $T$ T be a mapping on a bounded convex subset of a uniformly convex Banach space $X$, with $F(T) \neq \emptyset$. Assume that $T$ satisfies condition $(C^*)$. Define the sequence $x_n$ in $C$ by $x_1 \in C$ and $x_{n+1} = \lambda T x_n + (1-\lambda)x_n$ for $n \in \mathbb{N}$, where $\lambda \in (0,\ 1)$. Then (i) $\lim\{||x_n - q||\}$ exists, where $q \in F(T)$ (ii)$\lim_{n\longrightarrow\infty}\|Tx_n - x_n\| = 0$.

**Proof:** Let $q \in F(T)$. From the fact that $T$ satisfies condition $C^*$ and using lemmas (1), we have

$$\begin{aligned} ||x_{n+1}-q||^p &= ||\lambda(Tx_n - q) + (1-\lambda)(x_n - q)||^p \\ &\leq \lambda||Tx_n - q||^p + (1-\lambda)||x_n - q||^p - w_p(\lambda)g(||x_n - Tx_n||) \\ &\leq ||x_n - q||^p - w_p(\lambda)g(||x_n - Tx_n||^p) \end{aligned} \tag{7}$$

From (7) and lemma (2), $\lim\{||x_n - q||\}$ exists.
Also from (7), we have

$$\sum w_p(\lambda)g(||x_n - Tx_n||) \leq ||x_1 - q||^p$$

Since $g$ is continuous, strictly increasing and $g(0) = 0$, this yields $\lim_{n\longrightarrow\infty}\|Tx_n - x_n\| = 0$

**Theorem 5:** Let $T$ with $F(T) \neq \emptyset$, be a mapping on a compact convex subset $C$ of a uniformly convex Banach space $X$. Assume that $T$ is continuous and satisfies condition

$C^*$. Define a sequence $\{x_n\}$ in $C$ by $x_1$ in $C$ and $x_{n+1} = \lambda T x_n + (1-\lambda)x_n$ for $n \in \mathbb{N}$, where $\lambda \in (0,\ 1)$ is a real number. Then $\{x_n\}$ converges strongly to the fixed point of $T$.

**Proof:** Since $C$ is compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to $z \in C$. Since $\lim ||x_{n_j} - Tx_{n_j}|| = 0$ and $T$ is continuous, we have $\lim ||x_{n_j} - Tx_{n_j}|| = ||z - Tz|| = 0$. This implies $z \in F(T)$. From this, $\lim_{j \longrightarrow \infty} \|x_{n_j} - z\| = 0$, lemma (2) and lemma(4), the proof is complete.

**Theorem 6:** Let $T$ with $F(T) \neq \emptyset$, be a mapping on a nonempty, convex, closed and bounded subset $C$ of a uniformly convex Banach space $X$. Assume that $T$ is completely continuous and satisfies condition $C^*$. Define a sequence $\{x_n\}$ in $C$ by $x_1$ in $C$ and $x_{n+1} = \lambda T x_n + (1-\lambda)x_n$ for $n \in \mathbb{N}$, where $\lambda \in (0,\ 1)$ is a real number. Then $\{x_n\}$ converges strongly to a fixed point of $T$.

**Proof:** From lemma (4), $\lim ||x_n - Tx_n|| = 0$. Since $\{x_n\}$ is bounded and $T$ is completely continuous, then $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$which converges strongly. Hence $\{x_{n_k}\}$ converges strongly. Suppose $\lim\{x_{n_k}\} = z$. Then $\lim\{Tx_{n_k}\} = Tz$. Hence $\lim ||x_{n_k} - Tx_{n_k}|| = ||z - Tz|| = 0$, so that $z \in F(T)$. Lemmas (2) and (4) now imply $\lim ||x_n - z|| = 0$. This completes the proof of the theorem.

**Theorem 7:** Let $T$ with $F(T) \neq \emptyset$, be a mapping on a nonempty, convex, closed and bounded subset $C$ of a uniformly convex Banach space $X$. Assume $T$ is continuous, demi-compact and satisfies condition $C^*$. Define a sequence $\{x_n\}$ in $C$ by $x_1$ in $C$ and $x_{n+1} = \lambda T x_n + (1-\lambda)x_n$ for $n \in \mathbb{N}$, where $\lambda \in (0,\ 1)$ is a real number. Then $\{x_n\}$ converges strongly to a fixed point of $T$.

**Proof:** Since $T$ is demi-compact, then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges strongly. Let $\lim x_{n_k} = w$. Now, since $\lim ||x_{n_k} - Tx_{n_k}|| = 0$, and $T$ is continuous, we have $\lim ||x_{n_k} - Tx_{n_k}|| = ||w - Tw|| = 0$. This implies $w \in F(T)$. From $\lim ||x_{n_k} - w|| = 0$, lemma (2) and lemma (4), we have that $\{x_n\}$ converges to a fixed point of $T$.

**Theorem 8:** Let $C$ be a closed convex subset of *UCED* Banach space $X$. Let $S$ be a family of continuous demicompact commuting mappings on $C$ satisfying condition $C^*$. Then $S$ has a common fixed point.

**Proof:** Observe that $F(T_i) \neq \emptyset,\ \forall\ i \in S$. Further more by lemma (3), $F(T_i)$ is closed and convex for all $i \in S$. The rest of the proof now follows as in [22].

**Remark 1:** Observe that the mapping exhibited in our example is a continuous mapping. Hence the continuity condition imposed on $T$ in our theorems is natural.

**Competing Interests:** The authors declare that there are no completing interests surrounding this research work.

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